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The Propagation of a Crack by a Rigid Wedge in an Infinite Power Law Viscoelastic Body

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SUMMARY

Closed form solutions are obtained for the problem of a crack propagated by a 2-dimensional rigid wedge of finite thickness in an infinite power law viscoelastic body. The distance from the crack tip to the point of contact with the wedge is determined from the assumption that in the vicinity of the crack tip, forces of cohesion act to produce a smoothly closing crack with continuous stresses. Finally, a simple formula expressing the stress intensity factor as a function of the speed of penetration and the length of the free crack surface is exhibited.

INTRODUCTION

In a previous paper by Walton, Nachman and Schapery [1], hereafter referred to as WNS, simple closed form solutions were obtained for the 2-dimensional problem of a rigid indentor sliding over a power law viscoelastic halfspace. In this paper similar techniques are used to obtain an analytical solution to the problem of the propagation by a rigid wedge of a 2-dimensional crack in an infinite power law material.

This particular problem is an example of a physically realizable mode of fracture to which the correspondence principle does not apply. Moreover, the closed form solutions obtained here provide a method for experimentally determining physical parameters, such as the stress intensity factor, important for an understanding of fracture phenomena.

This problem has been studied extensively for elastic material, a detailed description of which appears in the article by Barenblatt [2]. The wedging problem was considered by Barenblatt for two reasons. In general, Barenblatt addressed himself to the issues of unrealistic singular crack tip stresses and indeterminant crack lengths encountered in fracture in classical linear elasticity. The wedge problem exhibits both features even though the crack is "semi-infinite" since the free crack surface ahead of the wedge is finite. In particular, the thrust of Barenblatt's article is that in a neighborhood of the crack tip, the separated surfaces experience cohesive molecular forces which produce a cusp profile and a finite stress field. Barenblatt accounts

for this by the phenomenological expedient of assuming that for equilibrium cracks a cohesive normal traction acts over a small interval behind the crack tip with a constant stress intensity factor characteristic of the material. As noted by Barenblatt, the finite crack length is determined by merely knowing this stress intensity factor and is independent of the actual distribution of cohesive forces. The utility of this observation lies in reversing the implication. Namely, if experimentally a wedge is driven into an elastic material and the length of the free surface of the resultant equilibrium crack is measured, then the stress intensity factor may be easily calculated.

Here we provided simple closed form expressions for the stress and displacement fields for the wedging of a power law viscoelastic material under the Barenblatt hypothesis. This analysis culminates in a relation among the material parameters, the material stress intensity factor and the length of the free crack surface.

Mathematically, the wedging problem reduces to a set of dual integral equations which in a sense are inverse to those considered in WNS. These equations are first solved without the Barenblatt hypothesis to simply see what the stress and displacement fields are for a power law viscoelastic material.

It was observed that the nature of the crack tip profile is dependent on the material whereas the nature of the stress singularity is not. In particular, the classical square root singularity in the

normal stress occurs for all materials, whereas crack tip profiles range from near square root (as in the elastic case) to cusp-like. Hence, singular stresses can occur even in the presence of a smoothly closed crack. With the addition of the Barenblatt hypothesis, the unknown crack length is then determined.

SECTION 1. FORMULATION OF THE PROBLEM.

We now consider the problem of the steady translation to the left (with velocity U) of a rigid 2-dimensional symmetric wedge with finite asymptotic thickness in an infinite power law viscoelastic body. (See Fig. 1.) Due to symmetry, we may consider the problem for the upper half plane only. The notation adopted here is consistent with that employed in WNS [1].

Neglecting inertia, the force balance equations are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0$$

and the boundary conditions are

$$\tau_{xy}(x, 0, t) = 0 \qquad -\infty < x < \infty$$

$$v(x, 0, t) = \begin{cases} 0 & x < Ut \\ f(x-Ut) & x > \ell_2 + Ut \end{cases}$$

$$\sigma_y(x, 0, t) = 0 \qquad Ut < x < \ell_2 + Ut.$$

Here $v(x, y, t)$ and $u(x, y, t)$ are the vertical and horizontal displacements, σ_x and σ_y are the normal stresses on x=constant

lines and y=constant lines respectively, τ_{xy} is the shear stress and $f(x)$ is the wedge profile whose finite asymptotic thickness is h, i.e. $\lim_{x \to \infty} f(x) = h$. The viscoelastic stress strain laws are

$$\sigma_x = \int_{-\infty}^{t} \Lambda(t-\tau) \frac{\partial \Delta}{\partial \tau} d\tau + 2 \int_{-\infty}^{t} G(t-\tau) \frac{\partial \varepsilon_x}{\partial \tau} d\tau$$

$$\sigma_y = \int_{-\infty}^{t} \Lambda(t-\tau) \frac{\partial \Delta}{\partial \tau} d\tau + 2 \int_{-\infty}^{t} G(t-\tau) \frac{\partial \varepsilon_y}{\partial \tau} d\tau$$

$$\tau_{xy} = \int_{-\infty}^{t} G(t-\tau) \frac{\partial \gamma xy}{\partial \tau} d\tau$$

where $\varepsilon_x = \frac{\partial u}{\partial x}$, $\varepsilon_y = \frac{\partial v}{\partial y}$, $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ and $\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ are the strains.

A detailed description of the next series of steps appears in WNS [1] and will be omitted here. After adopting the Gallilean variable $s = x + Ut$, all of the equations are differentiated with respect to s (save for the $\sigma_y(s, 0)$ boundary condition) and then Fourier transformed with respect to s. The resulting ODE's in y are solved and a Fourier inversion results in the following integral equations relating $v'(s, 0)$ to the normal traction $\sigma_y(s, 0)$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ips} (ip)^{-1} \overline{\left(\frac{\partial \sigma_y}{\partial s}\right)} dp = \sigma_y(s, 0) \qquad -\infty < s < \infty$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ips} A_1(p) dp = v'(s, 0) \qquad -\infty < s < \infty$$

where

$$\overline{\frac{\partial \sigma_y}{\partial s}}(s,\ 0) \;=\; 2(2\pi)^{1/2}(ip)|p|\ \frac{\overline{G}(\overline{\Lambda}+\overline{G})}{(\overline{\Lambda}+2\overline{G})}\ A_1(p).$$

(Here, $\overline{F}$ denotes the Fourier transform of F.)

To characterize our power law material it suffices to assume that

$$\frac{\overline{G}(\overline{\Lambda}+\overline{G})}{(\overline{\Lambda}+2\overline{G})} \;=\; \frac{\overline{E}}{4(1-\nu^2)}$$

where $E(t) = \mathcal{E}t^{-\alpha}H(t)$ and ν is Poisson's ratio which is taken to be constant. This leads finally to the dual relations

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A_1(p)e^{isp}dp \;=\; v'(s,\ 0) \qquad (1)$$

$$-\infty < s < \infty$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A_1(p)|p|(ip)^{\alpha-1}e^{isp}dp \;=\; g(s) \qquad (2)$$

where

$$g(s) \;=\; 2(1-\nu^2)\sigma_y(s,\ 0)/(U^{\alpha}\mathcal{E}\Gamma(1-\alpha)).$$

It should be noted that here $v'(s, 0)$ is known off of the compact interval $(0, \ell_2)$ on which $g(s)$ is known. Hence, the dual relations (1) and (2) are inverse to those derived in WNS [1].

The first problem considered is that of finding $v'(s, 0)$ ($= v'(s)$ for $0 < s < \ell_2$), ℓ_2 and ℓ_1, where $0 < \ell_1 < \ell_2$

and $f(\ell_1) = 0$, given that $g(s) = 0$ for $0 < s < \ell_2$. As will be observed later, this problem is underposed in that ℓ_2 (the length of the free crack surface) and ℓ_1 (the distance from crack tip to wedge point) cannot both be determined. Indeed, only an explicit value for the ratio ℓ_1/ℓ_2 will emerge.

To solve (1) and (2) we first rewrite (2) as

$$\frac{d}{ds}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A_1(p)|p|(ip)^{\alpha-2}e^{isp}dp = g(s). \qquad (3)$$

The representation of $A_1(p)$ obtained from (1) suggests the validity of the formal manipulations leading to line (3). The inversion of (1) and substitution of the resulting expression for $A_1(p)$ into (3) yields the following integral equation

$$\frac{d}{ds}\int_{-\infty}^{\infty} v'(y)K(s-y)dy = g(s) \qquad -\infty < s < \infty \qquad (4)$$

where

$$K(z) = -\frac{\Gamma(\alpha)}{\pi}\begin{cases}\cos(\alpha\pi)|z|^{-\alpha} & z > 0\\ |z|^{-\alpha} & z < 0.\end{cases}$$

The operator appearing on the left side of (4) is, in a generalized sense, inverse to the Abel type operator discussed in WNS [1]. This point requires some amplification. If K_1 denotes the operator of WNS [1] on say $(0, \ell_2)$, i.e.

$$K_1(f) = \int_0^{\ell_2} K_1(s-y)f(y)dy$$

$$K_1(z) = \frac{\Gamma(1-\alpha)}{\pi} \begin{cases} -\cos(\alpha\pi)|z|^{\alpha-1} & z > 0 \\ |z|^{\alpha-1} & z > 0, \end{cases}$$

and if K denotes the operator in (4) then Samko [3] proved that the product KK_1 is a weighted finite Hilbert transform which in a certain sense acts like an identity operator. On $(-\infty, \infty)$ K_1 and K are true inverses. Hence, if either f' or g is known on all of $(-\infty, \infty)$, the other may be easily determined. However, since here and in WNS [1] mixed boundary conditions are assumed, no simple inversion of the generalized Abel equations exists.

At this point it is necessary to observe that the nature of the wedging problem changes dramatically for $\alpha > 1/2$. Indeed, from the analysis of the asperity problem contained in WNS [1], it is apparent that the material will not remain in contact with a wedge of finite asymptotic thickness, but rather will leave off at some point ℓ_3 with $\ell_3 > \ell_2$. (See Fig. 1.) Consequently, the appropriate boundary conditions for $\alpha > 1/2$ are

$$v(s, 0) = \begin{cases} 0 & s < 0 \\ f(s) & \ell_2 < s < \ell_3 \end{cases}$$

$$\sigma_y(s, 0) = 0 \qquad 0 < s < \ell_2, \quad \ell_3 < s.$$

In this case, solving equation (4) is just as difficult as solving the multiple asperity problem mentioned in WNS [1], (Section 3). If the wedge profile is monotone increasing, this pathology does not occur for $0 \leq \alpha \leq 1/2$. Therefore, detailed solutions will be presented for this case only. Even though an explicit solution for $\alpha > 1/2$ cannot be exhibited, it will prove to be an easy matter to infer from the solution for $0 < \alpha < 1/2$ what the nature of the stress distribution and crack profile is in the vicinity of the crack tip when $\alpha > 1/2$.

Henceforth, we shall assume that $0 < \alpha < 1/2$, since the limiting cases $\alpha = 0$ (the classical elastic case) and $\alpha = 1/2$ may be then easily deduced. The case $\alpha = 1/2$ may also be treated directly in a simple manner, since the integral equation (4) reduces to an ordinary Abel equation.

To solve (4) when $0 < \alpha < 1/2$, we first integrate and recall that on $(-\infty, 0)$, $v'(s)$ vanishes and on (ℓ_2, ∞), $v'(s) = f'(s)$, a known function. Therefore we obtain the generalized Abel equation

$$\frac{\Gamma(\alpha)}{\pi}\left[\int_s^{\ell_2} \frac{v'(t)dt}{(t-s)^\alpha} + \cos(\alpha\pi)\int_0^s \frac{v'(t)dt}{(s-t)^\alpha}\right] = F(s) \qquad (5)$$

$$0 < s < \ell_2$$

where

$$F(s) = C - \frac{\Gamma(\alpha)}{\pi}\int_{\ell_2}^{\infty} \frac{v'(t)dt}{(t-s)^\alpha}$$

and C is an arbitrary constant of integration. Equation (5) is the equation considered in WNS [1] with α replaced by $1 - \alpha$. Its solution may be obtained either by the methods employed there or by that of Samko [3]. We adopt the former approach. However, the details of the derivation of the solution will be omitted since the essential ideas are contained in WNS [1].

We remark that for technical reasons a straight substitution into the formulas derived in WNS [1] cannot be used to solve the example considered in the next section since it results in divergent integrals. It proves more convenient to solve the corresponding Riemann-Hilbert boundary value problem in the class of functions unbounded at both $s = 0$ and $s = \ell_2$, thereby introducing an extra arbitrary parameter, K, which is later specified by imposing continuity conditions. The details will be omitted and only the final results presented.

It may be shown by the method employed in WNS [1] that $v'(s)$ must satisfy the ordinary Abel equation

$$\int_0^s \frac{v'(t)dt}{(s-t)^\alpha} = H_1(s) \qquad 0 < s < \ell_2 \tag{6}$$

with

$$\begin{aligned} H_1(s) &= Ks^{-1/2}(\ell_2-s)^{1/2-\alpha}\,\frac{B(\alpha,\, 1-\alpha)}{\alpha} \\ &- s^{-1/2}(\ell_2-s)^{1/2-\alpha}\,\frac{\Gamma(1-\alpha)}{\pi}\int_0^{\ell_2}\frac{\tau^{1/2}(\ell_2-\tau)^{\alpha-1/2}}{(\tau-s)}\,F(\tau)d\tau. \end{aligned} \tag{7}$$

The integral in (7) is a Cauchy principal value.

The function $v(s)$ is easily determined from (6) to be

$$v(s) = \frac{1}{B(\alpha,\ 1-\alpha)} \int_0^s \frac{H_1(t)dt}{(s-t)^{1-\alpha}} \tag{8}$$

where $B(\cdot,\ \cdot)$ denotes the beta function. The parameters K, C and ℓ_2/ℓ_1 are then chosen so as to insure the continuity of $v(s)$ at $s = 0,\ \ell_2$ and $v'(s)$ at ℓ_2.

In the next section, an example is considered for which a simple closed form solution is obtained for $v(s)$. We close this section with a useful observation. Once $v(s)$ is determined, $g(s)$ for $s < 0$ is easily calculated from (4). In particular, we have

$$g(s) = \frac{\Gamma(\alpha)}{\pi} \frac{d}{ds} \int_0^\infty \frac{v'(y)}{(y-s)^\alpha}\, dy. \tag{9}$$

It may be shown from (7) that K and C can be chosen so that

$$v(s) = As^{\alpha+1/2} + v_1(s) \tag{10}$$

where $v_1'(s)$ is continuous at $s = 0$. Hence for $0 \leq \alpha < 1/2$, $v(s)$ has a vertical tangent at the crack tip, $s = 0$, and from (9) it follows that $g(s) = O(|s|^{-1/2})$ as s approaches zero from the left.

It is apparent that (10) is also valid for $1/2 \leq \alpha < 1$. Consequently, for $\alpha = 1/2$ the free crack faces meet at a sharp corner and there results a square root singularity in the normal stress ahead

of the crack tip. However, for $\alpha > 1/2$, the crack surfaces form a cusp (i.e. close smoothly) with the normal stress still having a square root singularity. (See Fig. 1.) This is in marked contrast to the behavior of elastic materials.

SECTION 2. AN EXAMPLE.

To illustrate the technique of the previous section we choose a wedge shape, $f(s)$, for which the Cauchy integral in (7) is easily computed. Specifically, for $0 \leq \alpha < 1/2$ we take $f(s) = h[1 - (s/\ell_1)^{\alpha-1}]$, $s \geq \ell_1$. The fact that the wedge profile depends on α does not impose a serious limitation on the usefulness of the example, since the shape of $f(s)$ is qualitatively the same for all values of α.

With this choice of $f(s)$, the function $F(\tau)$ in (5) becomes

$$F(\tau) = C - \frac{\Gamma(\alpha)}{\pi} \ell_1^{1-\alpha} h\tau^{-1}[1 - (1-\tau/\ell_2)^{1-\alpha}].$$

The Cauchy integral in (7) may now be evaluated, and after some manipulation yields

$$\int_0^{\ell_2} \frac{\tau^{1/2}(\ell_2-\tau)^{\alpha-1/2}}{(\tau-s)} F(\tau)d\tau$$

$$= C\, B(\alpha+1/2,\ 1/2)\ell_2^{\alpha}(1-s/\ell_2)^{\alpha-1/2}{}_2F_1(1/2+\alpha,\ -1/2;\ 1/2;\ s/\ell_2)$$

$$- \lambda h \frac{\Gamma(\alpha+1/2)\Gamma(-1/2)}{\pi}(1-s/\ell_2)^{\alpha-1/2}{}_2F_1(1/2+\alpha,\ 1/2;\ 3/2;\ s/\ell_2) \qquad (11)$$

$$- \lambda h\Gamma(\alpha),$$

where the parameter λ is defined by

$$\lambda = (\ell_1/\ell_2)^{1-\alpha}. \tag{12}$$

After substitution of (11) into (7), the integral in (8) may be computed from which it follows that

$$v(s) = (K+\lambda h)\,\frac{B(\alpha,\,1/2)}{\pi}\,(s/\ell_2)^{\alpha-1/2}\,{}_2F_1(\alpha-1/2,\,1/2;\,\alpha+1/2;\,s/\ell_2)$$

$$-\,\frac{C\pi}{\Gamma(\alpha+1)}\,(s/\ell_2)^{\alpha-1/2}(1-s/\ell_2)^{1/2} \tag{13}$$

$$-\,\lambda h 2(s/\ell_2)^{\alpha-1/2}\,{}_2F_1(1/2,\,1/2;\,3/2;\,s/\ell_2).$$

For convenience, we define a new variable z by $z = s/\ell_2$ and consider $v(s)$ as a function of z, denoted $\hat{v}(z)$. To determine C, K and λ we impose continuity for $\hat{v}(z)$ near $z = 0, 1$ and for $\hat{v}'(z)$ near $z = 1$. It is readily seen that near $z = 0$ $\hat{v}(z)$ has the following form

$$\hat{v}(z) = C_1 z^{\alpha-1/2} + C_2 z^{\alpha+1/2} + 0(z^{\alpha+3/2}),$$

where

$$C_1 = B(\alpha,\,1/2)\,(K+\lambda h) - \frac{C\pi\ell_2^{\alpha}}{\Gamma(\alpha+1)} - 2\lambda h \tag{14}$$

and

$$C_2 = B(\alpha,\,1/2)\,\frac{(\alpha-1/2)}{2(\alpha+1/2)}\,(K+\lambda h) + \frac{C\pi\ell_2^{\alpha}}{2\Gamma(\alpha+1)} - \frac{\lambda h}{3}. \tag{15}$$

A similar analysis near $z = 1$ yields

$$\hat{v}(z) = D_1 + D_2(1-z)^{1/2} + O((1-z)^{3/2}) \tag{16}$$

with

$$D_1 = K \tag{17}$$

and

$$D_2 = (K+\lambda h)B(1/2,\ \alpha)(1-2\alpha) - \frac{C\pi}{\Gamma(\alpha+1)}\,\ell_2^{\ \alpha} + 2\lambda h. \tag{18}$$

Continuity of $\hat{v}(z)$ at $z = 0$ and $z = 1$ require that $C_1 = 0$ and $D_1 = f(\ell_2) = h(1-\lambda)$, and continuity of $\hat{v}'(z)$ at $z = 1$ follows from $D_2 = 0$. Hence, we obtain the relations

$$\frac{\pi C\ell_2^{\ \alpha}}{\Gamma(\alpha)} = \frac{\Gamma(\alpha+1)\Gamma(1/2)h(1-\alpha)}{\Gamma(\alpha+1/2)} \tag{19}$$

$$\lambda = \frac{\Gamma(3/2)\Gamma(\alpha+1)}{\Gamma(\alpha+1/2)}. \tag{20}$$

The constants K and λ are determined explicitly, whereas C and ℓ_2 occur coupled in relation (19). The addition of the Barenblatt assumption, discussed in the next section, will provide the required extra relation for the determination of C and ℓ_2.

It should be noted that the coefficient C_2 becomes

$$C_2 = \frac{2\Gamma(1/2)\Gamma(\alpha+1)(1-\alpha)h}{\Gamma(3/2+\alpha)\ 3} = \frac{4}{3}\lambda h \frac{(1-\alpha)}{(1/2+\alpha)} \tag{21}$$

which is clearly non-zero. This verifies line (10) in section 1 and hence the subsequent discussion on the nature of the singular normal stress.

SECTION 3. THE BARENBLATT HYPOTHESIS.

The Barenblatt hypothesis is that, for an equilibrium crack, forces of cohesion act on the separated crack surfaces within a small neighborhood of the crack tip to effect a smoothly closed crack and a cancellation of the singular normal stresses ahead of the crack tip. It is further assumed that the stress intensity factor corresponding to these cohesive forces is an intrinsic property of the material and, in particular, is independent of the length of the interval on which the forces act. The Barenblatt hypothesis is encorporated into the model discussed in section 2 by amending the stress free normal boundary condition as follows:

$$\sigma_y(s, 0) = p(s) \qquad 0 < s < \ell_2,$$

where $p(s)$ is some distribution of cohesive normal tractions which vanishes for $s \geq d$ and d is a small positive number. The tractions $p(s)$ are assumed to have a known stress intensity factor, L, given by

$$L = \frac{1}{\pi} \int_0^d \frac{p(s)}{\sqrt{s}}\, ds.$$

It is clear that the dual integral equations (1) and (2) are still valid. However, the generalized Abel equation (5) becomes

$$\frac{\Gamma(\alpha)}{\pi}\left[\int_s^{\ell_2} \frac{v'(t)dt}{(t-s)^\alpha} + \cos(\alpha\pi)\int_0^s \frac{v'(t)dt}{(s-t)^\alpha}\right] = F_1(s) \qquad (22)$$

$$0 < s < \ell_2$$

where

$$F_1(s) = F(s) - Q(s), \qquad (23)$$

$$Q(s) = -\int_s^d q(t)dt \qquad 0 < s < d \qquad (24)$$

and

$$q(s) = 2(1-\nu^2)p(s)/(U^\alpha E\Gamma(1-\alpha)). \qquad (25)$$

Equation (22) may be reduced, as before, to the ordinary Abel equation

$$\int_0^s \frac{v'(t)dt}{(s-t)^\alpha} = H_1(s) - H_2(s), \qquad (26)$$

where $H_1(s)$ is given by (7) and

$$H_2(s) = -s^{-1/2}(\ell_2-s)^{1/2-\alpha}\,\frac{\Gamma(1-\alpha)}{\pi}\int_0^{\ell_2} \frac{\tau^{1/2}(\ell_2-\tau)^{\alpha-1/2}}{(\tau-s)}\,Q(\tau)d\tau. \qquad (27)$$

Solving (26) we obtain, finally,

$$v(s) = \frac{1}{B(\alpha,\ 1-\alpha)}\left[\int_0^s \frac{H_1(t)dt}{(s-t)^{1-\alpha}} - \int_0^s \frac{H_2(t)dt}{(s-t)^{1-\alpha}}\right]. \qquad (28)$$

A few comments are now in order. As is well-known (Barenblatt [2]), it is the stress intensity factor, L, rather than the actual distribution of tractions, $p(s)$, that enters into an analysis of stresses at the crack tip and a determination of the free crack length, ℓ_2. Moreover, we shall show that, if d/ℓ_2 is small, then the displacement field is essentially independent of $p(s)$. Consequently, for illustrative purposes, we shall assume that $p(s)$ is equal to a constant, p, on $0 < s < d$.

Returning to the computation of $v(s)$, we denote the constant function $q(s)$ in (25) by q and observe that $Q(s)$ in (24) becomes

$$Q(s) = q(s-d) \qquad 0 < s < d.$$

Substitution of $Q(s)$ into (27) yields

$$H_2(s) = - s^{-1/2}(\ell_2-s)^{1/2-\alpha} q \frac{\Gamma(1-\alpha)}{\pi} \int_0^d \frac{\tau^{1/2}(\ell_2-\tau)^{\alpha-1/2}(\tau-d)}{(\tau-s)} d\tau. \qquad (29)$$

Due to the presence of $(\ell_2-\tau)^{\alpha-1/2}$ in the integrand, the Cauchy integral in (29) cannot be evaluated simply. Since we

ultimately will let d/ℓ_2 tend to zero, we may substitute $\ell_2^{\alpha-1/2}$ for $(\ell_2-\tau)^{\alpha-1/2}$ in (29).

Given this simplification, $H_2(s)$ becomes

$$H_2(s) = (s/d)^{-1/2}(1 - s/\ell_2)^{1/2-\alpha} q \frac{\Gamma(1-\alpha)}{\pi}$$

$$\cdot \begin{cases} d\, B(2, 1/2)\,{}_2F_1(-3/2, 1; 1/2; s/d) & 0 < s < d \\ -d(s/d)^{-1}B(2, 3/2)\,{}_2F_1(1, 3/2; 7/2; d/s) & d < s. \end{cases}$$

To analyse $v(s)$ in (28) we introduce some notation. Let $v_1(s)$ denote the displacement corresponding to a free crack surface, i.e. without the Barenblatt hypothesis. Then

$$v_1(s) = \frac{1}{B(\alpha, 1-\alpha)} \int_0^s \frac{H_1(t)dt}{(s-t)^{1-\alpha}} \qquad 0 < s < \ell_2$$

and is given explicitly by (13). Let $v_2(s)$ be given by

$$v_2(s) = \frac{1}{B(\alpha, 1-\alpha)} \int_0^s \frac{H_2(t)dt}{(s-t)^{1-\alpha}} .$$

As in the preceeding section, it is convenient here to define new variables $z = s/\ell_2$ and $\varepsilon = d/\ell_2$ and let $\tilde{v}$, $\tilde{v}_1$ and $\tilde{v}_2$ denote the corresponding functions of z and ε. Moreover, we introduce the parameter

$$L_1 = \frac{2qd^{1/2}}{\pi} = L\ 2(1-\nu^2)/(\nu^\alpha E\Gamma(1-\alpha))$$

where L is the stress intensity factor corresponding to the cohesive tractions $p(s)$ on $(0, d)$.

It is now a simple matter, albeit tedious, to substitute the expression for $H_2(s)$ into (28) and compute the necessary integrals, keeping in mind that $H_2(s)$ is defined differently on $(0, d)$ and (d, ℓ_2). However, our ultimate goal is the determination of ℓ_1 and ℓ_2 for which we need only consider the asymptotic form of $\tilde{v}(z)$ for z near 0 and 1. Moreover, we are assuming ε to be small, and hence shall combine all terms which vanish as $\varepsilon \to 0$ into a single residual term.

We first consider the asymptotics of $\tilde{v}(z)$ for z near zero, in particular we take $z < \varepsilon$. It may be shown that

$$\begin{aligned}\tilde{v}_2(z) &= (2L_1/3\Gamma(\alpha))[\varepsilon B(\alpha,\ 1/2)\ell_2^{\alpha+1/2} z^{\alpha-1/2}\,{}_2F_1(\alpha-1/2,\ 1/2;\ \alpha+1/2;\ z)\\ &\quad - 3B(\alpha,\ 3/2)\ell_2^{\alpha+1/2} z^{\alpha+1/2}\,{}_2F_1(\alpha-1/2,\ 3/2;\ 3/2+\alpha;\ z)\\ &\quad + \varepsilon^{1/2}\ell_2^{\ 1/2+\alpha}\int_0^z \frac{(t/\varepsilon)^{3/2}(1-t)^{1/2-\alpha}}{(z-t)^{1-\alpha}}\,{}_2F_1(1/2,\ 1;\ 5/2;\ t/\varepsilon)dt] \qquad 0 < z < \varepsilon\\ &= C_3 z^{\alpha-1/2} + C_4 z^{\alpha+1/2} + O(z^{\alpha+3/2}) \qquad (30)\end{aligned}$$

with

$$C_3 = (2L_1/3)\varepsilon \ell_2^{\alpha+1/2}\Gamma(1/2)/\Gamma(\alpha+1/2) \tag{31}$$

and

$$C_4 = 1/2C_3(\alpha-1/2)/(\alpha+1/2) - L_1\ell_2^{\alpha+1/2}\Gamma(1/2)/\Gamma(\alpha+3/2). \tag{32}$$

Since $\tilde{v}(z) = \tilde{v}_1(z) - \tilde{v}_2(z)$ we obtain for $0 < z < \varepsilon$

$$\tilde{v}(z) = (C_1-C_3)z^{\alpha-1/2} + (C_2-C_4)z^{\alpha+1/2} + 0(z^{\alpha+3/2}).$$

For finite stress we require

$$\begin{aligned} C_1 - C_3 &= 0 \\ C_2 - C_4 &= 0. \end{aligned} \tag{33}$$

A similar analysis must be performed for z near 1. Observe first that, if $\varepsilon < z < 1$,

$$\tilde{v}_2(z) = \tilde{v}_3(z) + \tilde{v}_4(z)$$

with

$$\tilde{v}_3(z) = \frac{\ell_2^{\alpha}}{B(\alpha,\ 1-\alpha)}\int_0^{\varepsilon} \frac{\tilde{H}_2(t)dt}{(z-t)^{1-\alpha}}$$

and

$$\tilde{v}_4(z) = \frac{\ell_2^{\alpha}}{B(\alpha,\ 1-\alpha)} \int_{\varepsilon}^{z} \frac{\tilde{H}_2(t)dt}{(z-t)^{1-\alpha}} \ .$$

Clearly $\tilde{v}_3(z)$ is smooth for z near 1 and is negligible for $\varepsilon << 1$. For $\tilde{v}_4(z)$ we obtain

$$\tilde{v}_4(z) = D_3 + D_4(1-z)^{1/2} + O((1-z)^{3/2})$$

with

$$D_3 = D_4 = O(\varepsilon). \tag{34}$$

To verify (34), we need only observe that

$$\tilde{v}_4(z) = (\ell_2^{1+\alpha} qB(2,\ 3/2)\varepsilon/\pi\Gamma(\alpha))$$

$$\cdot \int_{\varepsilon}^{z} \frac{(\varepsilon/t)^{3/2}(1-t)^{1/2-\alpha}}{(z-t)^{1-\alpha}} \ {}_2F_1(1,\ 3/2;\ 7/2;\ \varepsilon/t)dt.$$

Since $(\varepsilon/t)^{3/2} {}_2F_1(1,\ 3/2;\ 7/2;\ \varepsilon/t)$ is uniformly bounded for $\varepsilon \leq t \leq z \leq 1$, the estimate (34) is now evident.

For continuity of $\tilde{v}(z)$ and $\tilde{v}'(z)$ at $z = 1$ we require

$$D_1 - D_3 = f(\ell_2) = h(1-\lambda)$$

and (35)

$$D_2 - D_4 = 0.$$

Combining (31), (32), (33), (34) and (35), we arrive, finally, at the relations

$$K = h(1-\lambda) + O(\varepsilon),$$

$$\lambda = \Gamma(3/2)\Gamma(1+\alpha)/\Gamma(1/2+\alpha) + O(\varepsilon),$$

$$\frac{C\pi \ell_2^{\alpha}}{\Gamma(\alpha)} = \Gamma(1+\alpha)\Gamma(1/2)h(1-\alpha)/\Gamma(1/2+\alpha) + O(\varepsilon)$$

and

$$\ell_2^{\alpha+1/2} = -(2/3)h(1-\alpha)\Gamma(1+\alpha)/L_1. \tag{36}$$

We remark again that line (36) provides a means of experimentally determining the stress intensity for an equilibrium crack. Indeed, from (36) it follows that the stress intensity factor, L, associated with the cohesive tractions (which, under the Barenblatt assumption, is equal in magnitude but opposite in sign to the usual stress intensity factor for a free crack surface) is related to the wedge speed, U, and crack length, ℓ_2, by

$$L = -(1/3)h\Gamma(1+\alpha)\Gamma(2+\alpha)U^{\alpha}E\ell_2^{-\alpha-1/2}/(1-\nu^2).$$

Hence, for given speed U, the stress intensity factor L is easily computed once the length of the crack, ℓ_2, has been measured.

A few comments on the stress and displacement field under the Barenblatt hypothesis are in order. Outside of a neighborhood of order ε centered at the crack tip, the stress and displacement fields are merely a perturbation of order ε from the stress and displacement fields corresponding to a free crack surface (i.e. without a Barenblatt zone). It is only on an interval of order ε that the displacement profile suddenly becomes cusp-like and the stress distribution falls off rapidly to zero. In the limit $\varepsilon = 0$, (though keeping L constant) the stress and displacement fields are identical with those of a free crack, every where. However, unlike the problem posed with a free crack surface, we are still able to determine the crack length, ℓ_2, explicitly.

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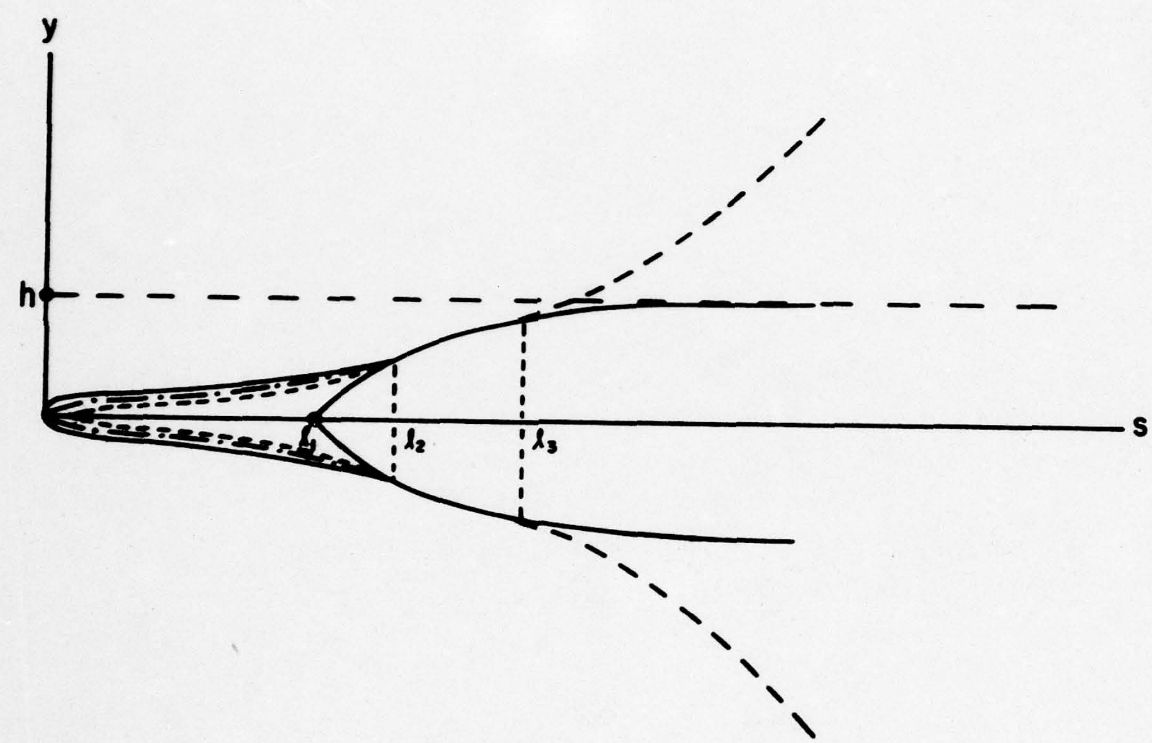
Figure 1. Displacement fields for $0 \leq \alpha < 1/2$ (————————), $\alpha = 1/2$(—•—•—•—) and $1/2 < \alpha < 1$ (-----).

Fig. 1



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Closed form solutions are obtained for the problem of a crack propagated by a 2-dimensional rigid wedge of finite thickness in an infinite power law viscoelastic body. The distance from the crack tip to the point of contact with the wedge is determined from the assumption that in the vicinity of the crack tip, forces of cohesion act to produce a smoothly closing crack with continuous stresses. Finally, a simple formula expressing the stress intensity factor as a function of the speed of penetration and the length of the free crack surface is exhibited.

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